Relaxation Algorithms for the Euler Equations

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Abstract

APPLIED to the linear scalar hyperbolic wave equation, the alternating direction implicit (ADI) central-difference scheme is unstable in three dimensions.[1] This is also true for the Euler equations in general.[2] Thus, the application of ADI schemes to three-dimensional Euler equations may encounter stability problems.

This paper shows how the ADI central-difference algorithms can be replaced by successive-line-relaxation (SLR) procedures that are stable in three dimensions. Several Beam- and Warming-type codes were modified using SLR solution procedures to compute flow over a cylinder: an NACA-0012 airfoil, and a shock wave reflecting off a plate.

Since Beam- and Warming-type codes are written in a "delta form," implementing relaxation procedures that operate only on the corrections is relatively easy; the steady-state residual computation and any artificial viscosity terms can be left unchanged.

Contents

SLR Applied to the Wave Equation

To illustrate the SLR procedure, the linear-scalar hyperbolic wave equation is used.

$$u_t + au_x + bu_y + cu_z = 0 \tag{1}$$

One observes that if the artificial viscosity term

$$\Delta x \Delta t \frac{|a|}{2} u_{xxt} \tag{2}$$

is added, the one-dimensional analog ($b = c = 0$) becomes

$$\left(I + \frac{\lambda_x}{2}\partial_x - \frac{|\lambda_x|}{2}\partial_{xx}\right)\Delta u = -\Delta t R^{n-1} \tag{3}$$

where $\lambda_x = a\Delta t/\Delta x$, $\Delta u = u^n - u^{n-1}$, and ∂ equals the undivided central differences. The residual R consists of the steady-state terms centrally differenced. This reduces to

$$(I + \lambda_x \partial_x^-)\Delta u = -\Delta t R^{n-1}, \qquad a > 0 \tag{4a}$$

$$(I + \lambda_x \partial_x^+)\Delta u = -\Delta t R^{n-1}, \qquad a < 0 \tag{4b}$$

where ∂^- and ∂^+ are first-order backward and forward differences. Thus, upwind difference operators are recovered for the correction Δu. The solution procedure for $a > 0$ involves a left-to-right sweep. For $a < 0$, a right-to-left sweep is used.

For the two-dimensional case, a similar artificial viscosity term is added, i.e.,

$$\left(I + \frac{\lambda_x}{2}\partial_x + \frac{\lambda_y}{2}\partial_y - \frac{|\lambda_x|}{2}\partial_{xx} - \frac{|\lambda_y|}{2}\partial_{yy}\right)\Delta u = -\Delta t R^{n-1} \tag{5}$$

For this case, the sweep direction is taken to be the x coordinate. For the left-to-right solution sweep, a tridiagonal equation is solved along each x = constant line. (See Fig. 1.) If a is negative, a right-to-left sweep is used.

For the general three-dimensional case, the SLR procedure may have two sweep directions (Fig. 2). For example, each step of a left-to-right sweep in the x direction is followed by a forward-to-backward (or vice versa) sweep in the z direction in the (y-z) plane.

Substituting the usual Fourier modes for the error for the left-to-right x sweep and the backward-to-forward z sweep, one finds all sweep directions to be stable.

Extension to the Euler Equations

The approaches were examined to extend the above idea to the Euler equations.

$$\partial_t Q + \partial_x E + \partial_y F = 0 \qquad \text{(Cartesian)} \tag{6}$$

To parallel the linear case, the maximum eigenvalues of the Jacobians $A = \partial E/\partial Q$ and $B = \partial F/\partial Q$ were used for the artificial viscosity terms. Since the Jacobians have multiple eigenvalues, the approach is only an approximation. Thus, in

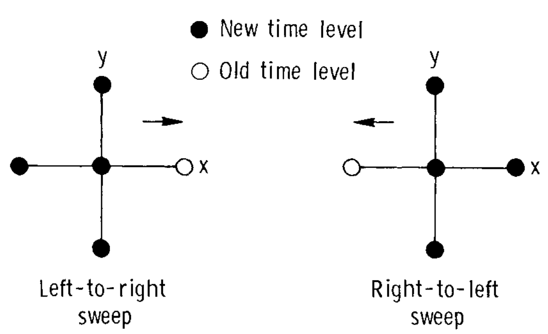

Fig. 1 SLR in two dimensions.

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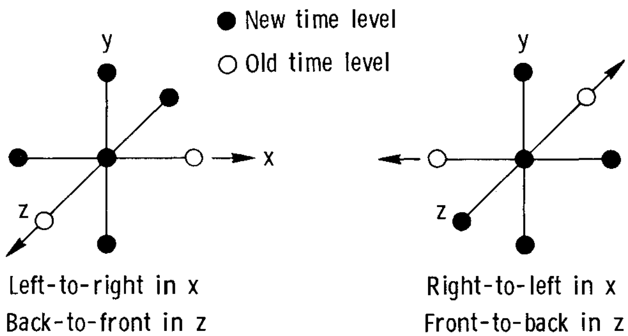
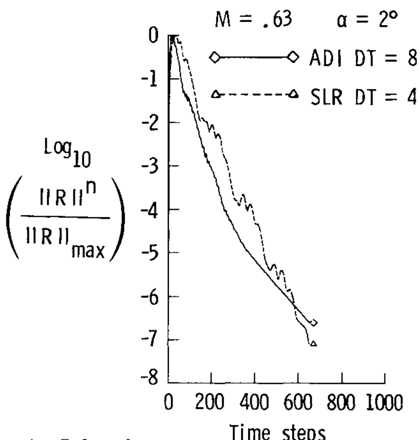
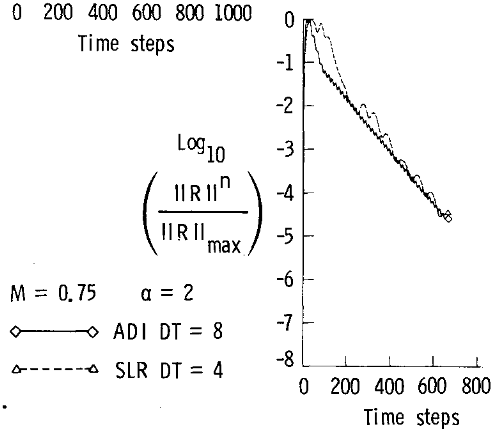


Fig. 2 SLR in three dimensions.



a) Subsonic.



b) Transonic.

Fig. 3 SLR/ADI convergence histories.

delta form, Eq. (7) is written as

$$(I+\Delta t\partial_x A+\Delta t\partial_y B-\nu_x\partial_{xx}-\nu_y\partial_{yy})\Delta Q=-\Delta t R^{n-1} \tag{7a}$$

where

$$\nu_x=\Delta t\,\frac{|u|+c}{2\Delta x}+\omega_x \qquad \nu_y=\Delta t\,\frac{|v|+c}{2\Delta y}+\omega_y \tag{7b}$$

with the sound speed denoted by c. The ω_x, ω_y constants correspond to implicit damping terms already existing in the codes.

Results and Discussion

Only the results for the airfoil are discussed here. These were obtained with the code of Pulliam et al.[3] using the SLR procedure, with the η coordinate being the marching direction. The grid was 128×32 cells in the ξ,η directions.

Figure 3 shows a comparison between the ADI/SLR convergence rates for subsonic and transonic cases. The SLR solution method shows a greater sensitivity to the initial phase. Once this initial phase is passed, the same approximate asymptotic rate is observed. This sensitivity may be related to the boundary conditions for the corrections. Although not attempted, it may be possible to increase the time step to the ADI value once the initial phase has passed.

Conclusions

It has been demonstrated that SLR solution procedures can be applied to the Euler equations using a central-difference scheme for the residual, while the correction operator approximates roughly upwind differences. Although all the test cases presented are two-dimensional, the extension to three dimensions is stable.

References

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[3]Pulliam, T. H., Jespersen, D. C., and Childs, R. E., "An Enhanced Version of an Implicit Code for the Euler Equations," AIAA Paper 83-0344, Jan. 1983.